DOCUMENT RESUME

ED 143 531 SE 023 009

AUTHOR	Anderson, R. D.; And Others
TITLE	Mathematics for Junior High School, Supplementary Units. Commentary for Teachers. Revised Edition.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	60
NOTE	65p.; For related document, see SE 023 008; Contains occasional light and broken type
EDRS PRICE	MF-$0.83 HC-$3.50 Plus Postage.
DESCRIPTORS	*Algebra; *Geometry; Junior High School Students; *Mathematics; Mathematics Education; Number Concepts; Secondary Education; *Secondary School Mathematics; *Teaching Guides
IDENTIFIERS	*School Mathematics Study Group

ABSTRACT

This is the Teacher's Commentary for the Supplementary Units for Junior High School Students. Included in the Commentary are background material for teachers, suggestions for instruction, and answers to student exercises. Also included are comments on how to use the materials with different types of students and time needed for instruction. (RH)

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SCHOOL
MATHEMATICS
STUDY GROUP

MATHEMATICS FOR JUNIOR HIGH SCHOOL

SUPPLEMENTARY UNITS

Commentary for Teachers

(revised edition)

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MATHEMATICS FOR JUNIOR HIGH SCHOOL

SUPPLEMENTARY UNITS

Commentary for Teachers

(revised edition)

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Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

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COMMENTARY FOR TEACHERS

SUPPLEMENTARY UNIT 1

SETS

This is a supplementary chapter on sets. Although it is made available with the more adept pupils in mind, the average pupil should be able to follow the development. It should be possible to cover the material in six to eight (50 minute period) lessons, including a test.

It would also be possible to let a few bright students work through this unit on their own time. When used this way, the unit should require a minimum of teacher help.

The idea of "set" has been used rather extensively in our work, beginning with Chapter 2 of the seventh grade. The general structure of sets, however, has not been extensively developed. In view of the importance and usefulness of set theory in mathematics, especially in the understanding of algebra, it is thought desirable to include this chapter for possible use and enrichment.

Because of the way in which the text is developed, it is not necessary to have much additional material in this commentary.

1-1. Introduction

This section is in the nature of a review. We explain that a set is merely a collection of objects, any objects, and calls the pupils' attention to the fact that the idea of "set" has been used elsewhere.

1-2. Sets, Their Number and Their Subsets

This section begins by explaining further the concept of set. Following this, sets are defined and discussed from the standpoint of their members, or elements, of a set. Notice especially the paragraph headed "Property." Be sure the symbol "{ }" is understood and properly used.

Exercises 1-2-a. Answers

1. (a) Names of girls.

   (b) U. S. Presidents and Generals.

   (c) The odd counting numbers.

   (d) Multiples of 2, 4, 6, 12.

2. (a) Tom is a member of the set whose members are Carl, Jim, Tom and Robert.

   (b) 6 is a member of the set of all even whole numbers.

   (c) If X is a member of the set whose members are Tom, Carl, Bob, and Jim, then X must be equal to Tom, or equal to Carl, or equal to Bob, or equal to Jim.

3. (a) True (c) False (e) False (Washington, D.C. is not a state.)

   (b) True (d) True

4. (a) {2, 3, 6}

   (b) {violin player, viola player, cello player}

   (c) {0, 1, 2, 3, 4, 5, 6, ...}

   (d) {Hoover, Roosevelt, Truman, Eisenhower}.

Always feel free to introduce additional problems, especially problems which will interest the pupils.

Subsets

A set R is a subset of a set S if every element of R is an element of S. It is important that the pupils understand this definition and the use of the symbols, "$\subset$" and "$\supset$". The pupils should enjoy the Venn diagrams. Perhaps you will want to introduce more examples of diagrams. See page 12 for null set.

Exercises 1-2-b. Answers

1. (a) If X is a member of the set of all red flowers, then X is a member of the set of all flowers.

   (b) M is contained in N, and N contains M.

(c) The set of the odd counting numbers is contained in the set of all counting numbers.

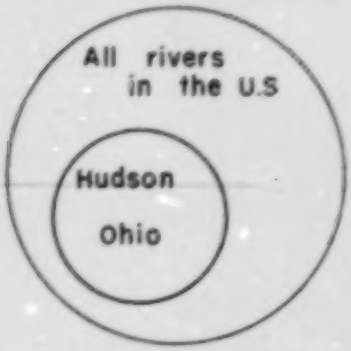
2. {4}, {5}, {6}, {4,5}, {4,6}, {5,6}, {4,5,6}.

3. (a) {12, 20, 32} $\subset$ {0, 1, 2, 3, 4, ...}.

(b) {Great Lakes} $\supset$ {Lake Huron, Lake Michigan}.

(c) {Hoover, Truman} $\subset$ {Wilson, Harding, Coolidge, Hoover, Roosevelt, Truman, Eisenhower}. or
{Hoover, Truman} $\subset$ {All U. S. Presidents since 1920}.

4. (a)



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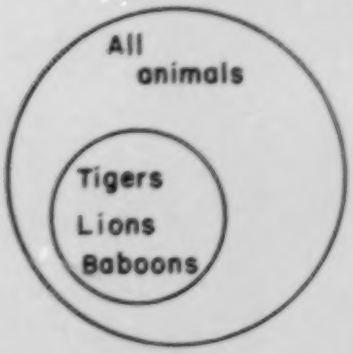

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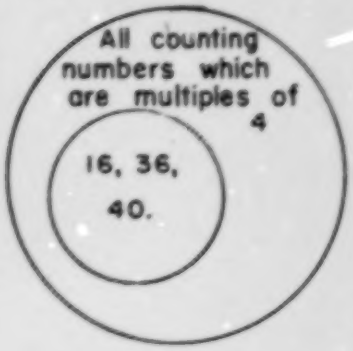

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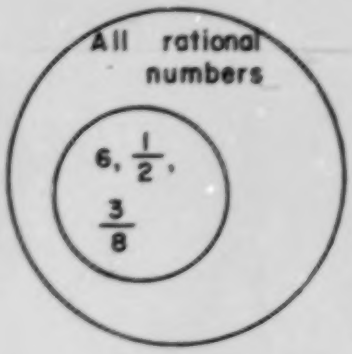


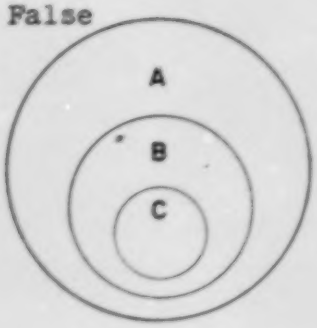
5. (a) False

(b) False

(c) True

(d) True

6. Yes



1-3. Operations with Sets

In this section the operations (union and intersection) with sets are introduced. Also the definition of equality (identical sets) is given. In addition the commutative, associative, and distributive properties are discussed. The similarity in the use of these properties with sets and with the rational numbers should be discussed.

Exercises 1-3-a. Answers

1. (a) $M \cup N = \{\text{Red, Blue, Green, Yellow, White}\}$.

   (b) Yes. Commutative property.

2. (a) Yes. $(A \cup B) \subset C$, and $C \subset (A \cup B)$

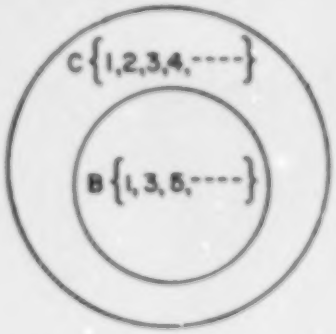
   (b) Yes. A is a subset of C.

   (c) No. A and B have no elements in common.

   (d) Yes. Commutative property.

   (e) No. A and B have no elements in common.

   (f)



   (g) No. A is not contained in B and B is not contained in A.

3. (a) Yes. Associative property.

   (b) Yes. The definition of equality of sets.

4. (a) True. Since X is a subset of S, and is, therefore, a member of S.

   (b) True. C is a subset of S.

   (c) False. $C \subset S$, but S is not contained in C.

(d) False. S is not a subset of C.

(e) True. X is a subset of C.

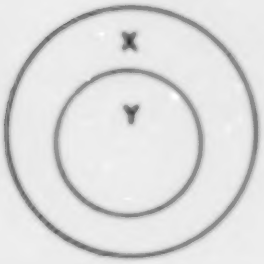
(f) True. C is a subset of S.

(g) Yes, since $X \in C$.

Yes, since $X \in S$.

(h) Yes, since $S \subset C$.

5. (a)



(b) $Y \subset X$. Y is a subset of X.

Exercises 1-3-b. Answers

1. (a) $A \cap B = \{girl, chair\}$.

(b) $A \cap C = \{chair\}$.
$C \cap A = \{chair\}$.

(c) $A \cap (B \cup C) = \{boy, girl, chair\} \cap (\{girl, chair, dog\} \cup \{chair, dog, cat\}) = \{boy, girl, chair\} \cap \{girl, chair, dog, cat\} = \{girl, chair\}$.

$(A \cap B) \cup (A \cap C) = (\{boy, girl, chair\} \cap \{girl, chair, dog\}) \cup (\{boy, girl, chair\} \cap \{chair, dog, cat\}) = \{girl, chair\} \cup \{chair\} = \{girl, chair\}$.

(d) $A \cap (B \cup C) = \{boy, girl, chair\} \cap (\{girl, chair, dog\} \cap \{chair, dog, cat\}) = \{boy, girl, chair\} \cap \{chair, dog\} = \{chair\}$.

$C \cap (A \cup B) = \{chair, dog, cat\} \cap (\{boy, girl, chair\} \cap \{girl, chair, dog\}) = \{chair, dog, cat\} \cap \{girl, chair\} = \{chair\}$.

2. (a) Yes. Commutative property.

   (b) Yes. ϕ is a subset of H.

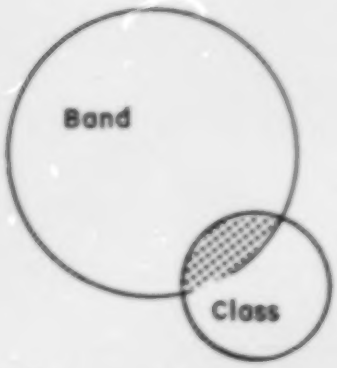
   (c) Identity element.

3. (a) Parallel.

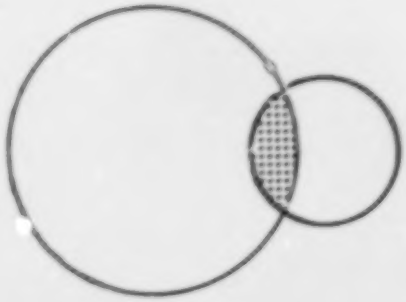
   (b) They intersect.

4. Yes. The commutative, associative and distributive properties apply to both sets.

5.



6.



7. (a) Odd counting numbers.

   (b) ϕ

8. (a) Yes. A is a subset of B.

   (b) Yes. The intersection of two sets contains only those elements which are common to both sets.

1-4. Order, One-to-One Correspondence

The first concept we will consider is that of order. As pointed out in the text, ordered sets are very important in many branches of mathematics. The concept is also important in other fields of science, as well as many other areas. For example, ordered pairs are most important in a shoe store.

The second concept is a basic one, that of one-to-one correspondence. The ideas developed here should be thoroughly understood by the pupils.

The possible number of matchings is in reality, a permutation problem. Since some of the students that will use this unit may not have studied the chapter on permutations, an alternate method for finding the number of possible matchings is developed in the exercises. The method given requires some experimenting to find how many times one pair is matched when all permutations are used. Students who are familiar with factorial numbers should be encouraged to discover this relationship rather than the one given.

Exercises 1-4. Answers

1.

Set A	Set B
Bill	Jane
Tom	Ann
Sam	Susan

Set A	Set B
Bill	Susan
Tom	Ann
Sam	Jane

Set A	Set B
Bill	Jane
Tom	Susan
Susan	Ann

Set A	Set B
Bill	Susan
Tom	Jane
Susan	Ann

2. (a) 3

(b) 2

(c) Multiplication, $3 \cdot 2 = 6$

3. (a)

1↔a	1↔a	1↔a	1↔a	1↔a	1↔a
2↔b	2↔b	2↔c	2↔c	2↔d	2↔d
3↔c	3↔d	3↔b	3↔d	3↔b	3↔c
4↔d	4↔c	4↔d	4↔b	4↔c	4↔b

(b) 6

(c) 6

(d) 6

(e) 6

(f) 24

*(g) Multiply the number of times that one pair is matched in all possible matchings by the number of elements in each set. Or, if the student has progressed this far, $n!$, where n is the number of elements in one set.

4. (a) True. For example: $\{7,8,9\} = \{7,9,8\}$, and also $\{7,8,9\} \longleftrightarrow \{7,9,8\}$.

(b) False. For example: $\{1,2,3\} \neq \{7,8,9\}$, but $\{1,2,3\} \longleftrightarrow \{7,8,9\}$.

(c) True. See example for Part (a).

(d) False. See Part (a).

5. (1, x)
(2, y)
(3, t)
(4, a)
(5, b)
(6, c)

6. No. One-to-one correspondence between the set of eggs and set of receptacles in the carton.

7. Yes. For every element in X there is a corresponding element in y, and for every element in y, there is a corresponding element in X.

8. No. There is a one-to-one correspondence between the two sets of points.

1-5. The Number of a Set and Counting.

The section devoted to the number of a set, and to counting is basic. These topics should be interesting and should increase the pupils' experience in working with sets.

Exercises 1-5. Answers

1. (a) 6. (c) 5. (e) a and d.
   (b) 4. (d) 6.
2. $R \leq S$.
3. M matches R.
4. The same number.
5. C_{12}: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}.
   C_7: {1, 2, 3, 4, 5, 6, 7}.

C_{12}	C_7
1 ⟷	1
2 ⟷	2
3 ⟷	3
4 ⟷	4
5 ⟷	5
6 ⟷	6
7 ⟷	7
8 ⟷	
9 ⟷	
10 ⟷	
11 ⟷	
12 ⟷	

6. $A \cup B$ = {Bob, Sue, Tom, Joe, cat, dog, chair}.

   C = {1, 2, 3, 4, 5, 6, 7, 8, 9, ...}.

   $A \cup B = C_7$.

7. $M \cup N$ = {1, 2, 3, 4, 5, 6, 7, 8, 9}.

   C = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ... }.

   $M \cup N = C_9$.

SUPPLEMENTARY UNIT 2

SPECIAL FIGURES IN PROJECTIVE GEOMETRY

This brief unit is a little introduction to projective geometry. The basic idea is a change in language which replaces "two lines are parallel" by "two lines intersect in an ideal point." It should be emphasized that this is just a change in language and does not mean that two parallel lines intersect in a point "way out there." However, the change is a fruitful one because with this interpretation we can say that "any two lines determine a point." This is the basis of the "principle of duality" in projective geometry which is only hinted at in the text. Roughly speaking it means that if a theorem about a plane configuration composed of lines and points is true, then its "dual" theorem in which the words "line" and "point" are interchanged is also true. The Desargues' Theorem and its converse are dual theorems. You will note the duality in the statements about committees just preceding Exercises 2-3.

Desargues' Theorem, beside being one of the basic theorems of projective geometry is very interesting in itself and the pupils should get much enjoyment out of drawing various figures showing this configuration. Problems 2 and 3 in Exercises 2-3 are especially fruitful in this connection.

This unit depends on very little of the content of Junior High School Mathematics, Vol. 1. Pupils would probably enjoy it more after they have studied Chapter 4, than they would before their introduction to non-metric geometry.

The early chapters of the Carus Monograph by John Wesley Young which is entitled "Projective Geometry" would be a very useful teacher reference.

Exercises 2-1. Answers

1. Here the ideal point of one line corresponds to the ideal point of the other.
2. One-to-one correspondence.
3. The line $\overleftrightarrow{PQ}$.
4. The line through P parallel to $\overleftrightarrow{AB}$. This line intersects ℓ in its ideal point.
5. This means: There is just one line through P parallel to ℓ
6. (a) Ideal point.
   (b) To get the point on ℓ_1 corresponding to G', draw a line through G' parallel to the four parallels and cutting ℓ_1 in G. A similar construction gives the other two.

Exercise 2-2

The answer is a drawing which is illustrated in the text.

Exercises 2-3. Answers

1. Figures are called for.
2. $\overleftrightarrow{CO}$ and $\overleftrightarrow{QA'}$ intersect in C'; $\overleftrightarrow{OB}$ and $\overleftrightarrow{A'P}$ intersect in B'; $\overleftrightarrow{BC}$ and $\overleftrightarrow{PQ}$ intersect in R. C', B', and R are collinear.
3. Choose carefully three points for the vertices of the first triangle. One choice would be AQA'. Then the second triangle would have to be BRB'. Then $\overleftrightarrow{AQ}$ and $\overleftrightarrow{BR}$ intersect in C; $\overleftrightarrow{QA'}$ and $\overleftrightarrow{RB'}$ intersect in C'; $\overleftrightarrow{A'A}$ and $\overleftrightarrow{B'B}$ intersect in O. Then C, C', and O are collinear.
4. Actually the same kind of figure would appear but it would be drawn differently.

5. This is really a matter of verification. This configuration is called a "finite geometry" - it consists of 7 points and 7 "lines." (See B. W. Jones "Miniature Geometries" The Mathematics Teacher, vol. 52 (1959) pp. 66-71).

SUPPLEMENTARY UNIT 3

REPEATING DECIMALS AND TESTS FOR DIVISIBILITY

3-1. Introduction

This unit is for unusually good students. One of the purposes of this monograph is to encourage the student to be self-reliant. The teacher should only give him as much help as is necessary to keep him going and refuse help when it becomes apparent that the student wants the teacher to save him mental effort. The teacher might even want to make application of these principles himself and see how much he can do without the help of this commentary.

3-2. Casting Out the Nines

Here the student should see the advantage of multiplying the numbers "algebraically" using the distributive property and commutative property. Probably one of the hardest things to see in this section is that if $9b + r$ is divided by 9, the remainder is the same as when r is divided by 9. Fundamentally the explanation is this: Every number can be divided by 9 yielding a quotient, q, and a remainder r. Thus if N is the number, it can be written $9q + r$ where r is the remainder, that is, is zero or a natural number less than 9. Conversely, if a number is written in the form $9q + r$ where r is a natural number less than 9, then r is the remainder when the number $9q + r$ is divided by 9. Thus what the expansion of 156,782 in powers of ten shows, is that it can be written in the form $9q + (1 + 5 + 6 + 7 + 8 + 2)$. Then the sum in parentheses can be written in the form $9t + 2$, where 2 is the remainder after division by 9. Hence 156,782 is equal to $9(q + t) + 2$ which means that 2 must be the remainder when the sum of the digits is divided by 9. This is also dealt with in the students' material. The student can probably see this better in terms of numbers, although it may be easier for the teacher to see in letters. While the student should eventually

get to the use of letters, the important thing is for him to see what is happening. This can probably best be done by working out a number of numerical problems along the lines given in the text.

1. (a) 226843, 67945, 45654 are not divisible by 9.
427536 is divisible by 9.

(b) The remainder upon division by 9 is equal to the remainder when the sum of the digits is divided by 9.

(c) see (b) above.

In Exercise 2, the student might write $69 + 79 = 7 \times 9 + 6 + 8 \times 9 + 7 = 7 \times 9 + 8 \times 9 + 6 + 7 = (7 + 8) \times 9 + 6 + 7$. He should be required to name what properties of numbers (commutative, associative and distributive properties) he is using. This is a good opportunity to impress these properties on him, but it should not be done to the point of boredom. When he gets to the point that it is clear that he knows what is going on, he should not be required to name the property each time. For a sum of three he could show from first principles or, by combining terms, reduce it to summing two things twice just as $a + b + c = (a + b) + c$.

In Exercise 3, the student would of course write 69 as $9 \times 7 + 6$ and 79 as $11 \times 7 + 2$. In Exercise 4, one would proceed as above except that one would multiply instead of add. The same results would hold for 23 since one could write 79 as $3 \times 23 + 10$ and 69 as $3 \times 23 + 0$. They would hold equally well for any number.

The general principle should be formulated as soon as the student can do so himself. The remainder when the sum (or product) of any two numbers is divided by a given number, is the same as when the sum (or product) of the remainders is divided by a given number.

In Exercise 5 the answer is 1^{20}, which is 1 whether the divisor is 99 or 3. If the divisor is 99, we can write 10^{20} as 100^{10} and see that then the remainder will be 1^{10} which is also 1. The bright student might want to know what would happen if the divisor were 999. Then one would write 10^{20} as $1000^{6} \times 100$ and so the remainder would be 1×100 which is 100. By a similar argument the remainder when 7^{20} is divided by 6 is 1.

The student might want to carry this farther and consider the remainder when 10^{20} is divided by 7. This could be worked out as follows: 10^{20} could be replaced by 3^{20} which is equal to 9^{10}. That remainder would be the same as that for 2^{10} which is $(2^3)^3 \times 2$ or $8^3 \times 2$ which has the remainder $1^3 \times 2$ or 2. A student bright enough to be interested in developing this should have no trouble with manipulating the exponents. He might want to explore what would be the last digit in the huge number 3^{40}. This would be just the remainder when 3^{40} is divided by 10. Since 3^4 has a remainder of 1 when divided by 10, the answer would be 1^{10} or 1. This exercise might also be done by looking at the pattern of last digits in the powers of 3: 3, 9, 27, 81. The last digits form the pattern 3, 9, 7, 1, 3, 9, 7, 1 and so forth. These things could be explored still further.

For Exercise 7 the simplest test for divisibility by 4 is to test whether the number consisting of the last two digits is divisible by 4. For instance, 178524 is divisible by 4 since 24 is. This is because 178524 is equal to 178500 + 24 and any multiple of 100 is divisible by 4. Similarly to test for divisibility by 8 one uses the last three digits. Another test for divisibility by 4 would be to see that if the last digit is 0, 4 or 8 the number is divisible by 4 if the next to the last digit is even. If the last digit is 2 or 6, the number is divisible by 4 if the next to the last digit is odd. The reasons should of course be found.

For Exercise 8, the number written to the base twelve would be a multiple of twelve plus the last digit. Thus the number would be divisible by any divisor of 12 if the last digit is

divisible by this divisor. That is, if the number is divisible by 6, its last digit must be divisible by 6. Other numbers in place of 6 would be 2, 3, 4, 12. If the base were 7, since 7 has no factors but itself and 1, the only number that could correctly go in the blank would be 7.

Exercise 9 is rather fundamental but requires some insight on the part of the student. Just as a decimal terminates when the divisor is a divisor of a power of 10, so a "decimal" to be base 7, will terminate when the divisor is a divisor of a power of 7. That is, $\frac{1}{7}$ and $\frac{1}{49}$ will have terminating "decimals" to the base 7. The decimal expansion of a fraction depends on the number base. But whether or not it is rational is independent of the base. A number that is a prime number when expressed in the decimal notation is a prime number when expressed to any other base, since the property of being prime is a property of the number and not of the way in which it happens to be written. You may not want to tackle this point at this stage with your students, but you perhaps should be prepared to meet it if it occurs.

The answer to the **Exercise 10** would be: the remainder when $2 + 2 + 2 + 2 + 2$. or 10, is divided by 7. Hence the remainder is 3. **Exercise 11** is easy and the answers to **Exercises 12** and 13 are developed in the text after the exercises.

In what follows it is probably wise to deal with the discussion with letters since here the advantage is quite clear and the manipulation is not very complex.

Probably the students will be interested in doing quite a little casting out of the nines in numerical examples. There is an amusing little book by E. T. Bell on "Numerology" which the students might like to read.

In the next set of exercises, the justification of the casting out of nines is that the remainder when the sum (or product) of two numbers is divided by 9 is the same as the remainder when the sum of the digits is divided by 9, as well as the

properties described in Exercises 2 and 4 of the previous sets. For instance, suppose a product ab is to be checked. The remainder when ab is divided by 9 is the same as the remainder when the product of their remainders is divided by 9 which is the same as the remainder when the product of the sums of their digits is divided by 9. So whichever way one does it, he is computing the remainder after division by 9 -- it does not matter anywhere whether a number or the sum of its digits, is divided by 9 as far as the remainder goes. The check of course only checks the remainder after division by 9. Thus when 810858 or any other product differs from 809,778 by 9 times any number, we have an incorrect product that will still check.

For Exercise 4, the remainders would be 3 for division by 7, and, since the remainder when 7 is divided by 6 or 3 is 1, the remainder when the number is divided by 6 or 3 would be the same as when the sum: 5 + 3 + 2 + 1 + 4 + 3 is divided by 6 or 3. The sum is equal to 18 which is divisible by both 6 and 3. Hence the given number is divisible by 6 and 3, that is, has a remainder 0 when divided by 6 or 3.

The short-cuts asked for in Exercise 5 are described in the text that follows. As Exercise 4 shows, one would cast out threes, or sixes or twos in the numeration system to the base 7, since 6, 2, and 3 are divisors of 7 - 1 just as 3 and 9 are divisors of 10 - 1.

Exercise 6. In a numeration system to the base 7, casting out sixes would have a result corresponding to that in the decimal system when nines are cast out.

For Exercise 7, it may be seen that scrambling the digits does not alter the value of their sum nor the remainder when the numbers are divided by 9. Hence each of the numbers is of the form 9n + r and 9t + r, with the remainders the same. Then

if we subtract one from the other, we get $9n + r - 9t - r$ which is equal to $9(n - t)$, that is a multiple of 9. Thus the sum of its digits is a multiple of 9. When the sum of the digits given is a multiple of 9, one cannot be sure whether the missing digit is 9 or 0. Otherwise the trick can be worked by adding the sum given and seeing what number added to this sum will give a multiple of 9. For instance, in the example given, one must add 4 to 14 to get a multiple of 9 and hence 4 is the missing digit.

3-4. Divisibility by 11

The exercises are solved in the text which follows them. You may have a little trouble with the product of - 1 and - 1 and you may want to avoid this; this can be done in the example worked out by seeing that 10^3 is $10^2 \times 10$ which would have a remainder of $1 \times (-1)$ or - 1. If you wish to avoid negative numbers completely, you could confine yourself to the first test. Or you could show the test for two-digit numbers by noticing that if the second digit is larger than the first one, the remainder after division by 11 is the second digit less the first one (e.g., $79 = 77 + 2$). If the second digit were smaller than the first one, one could add 11 to it and subtract the first digit, giving a correct remainder (e.g., for 73, subtract 7 from $3 + 11$ and see that the remainder is 7). This is somewhat laborious, however.

It is important in testing for the remainder after division by 11 by the second method to start at the right hand end of the number. If you are merely testing for divisibility it does not matter at which end you start.

For Exercise 2, page 54, the test for divisibility by 8 would be analogous to that for 11 in the decimal system.

For Exercise 3, the number 157,892 is equal to 157(1000) + 892 or 157 × 999 + (157 + 892). Hence if d is any divisor of 999, the remainder when 157,892 is divided by d is the same as when 157 + 892 is divided by d. The prime divisors of 999 are 3, 37. In fact $999 = 3^3 \times 37$. This test would work for any divisor of 999, that is, for 3, 9, 27, 37, 3 × 37, 9 × 37 as well as, of course, for 999 itself.

The students might also be interested in writing 157,892 as equal to 157(1001) + 892 - 137. Then if d is any divisor of 1001, the remainder when 157,892 is divided by d is the same as when 892 - 157 is divided by d. The prime divisors of 1001 are 7, 11, and 13. This would give a test for divisibility by 7 and 13 as well as 11. This has some connection with the fact that the decimal equivalents of $\frac{1}{7}$ and $\frac{1}{13}$ have six digits in the repeating portion. This can be developed further.

For Exercise 4, consider a number $534{,}623_{seven}$. Then in the numeral system to base seven, 534,623 is equal to $534_{seven} \times 666_{seven} + 534_{seven} + 623_{seven}$. Hence we would have to find the divisors of 666_{seven}. Now $666_{seven} = 6_{seven} \times 111_{seven}$. To see if 111_{seven} has factors it is probably easiest to convert to the base ten. So $111_{seven} = 49 + 7 + 1 = 57$ which has 3 and 19 as its factors. Actually we could have tested 111_{seven} for divisiblity by 3 by adding the digits since this works for 3 in the system to the base seven as well as to the base ten. So in the numeration system to the base seven, this kind of test would work for 19, that is 25_{seven} as well as for 2, 3, 6 which are written the same way in both systems.

Similarly for the numeration system to the base twelve, we would seek the divisors of eee_{twelve} which is $e_{twelve} \times 111_{twelve}$ and e stands for "eleven." We cannot say that 111_{twelve} is divisible by 3 since 3 is not a divisor of one less than the base. But we have to convert it to the decimal notation. It is 144 + 12 + 1 = 157, which is a prime number. Hence this kind of test would work for 3 and 157 in the decimal system or for 3_{twelve} and 111_{twelve}.

The connection between repeating decimals and tests for divisibility should begin to emerge for the student here. Suppose m is some number which is divisible by neither 2 nor 5. Then some power of 10 is 1 more than a multiple of m. This can be seen as follows. We know that the remainders when the powers of 10 are divided by m are the numbers from 1 to $m - 1$ inclusive. Since there are infinitely many powers of 10, two of the powers must have the same remainder, that is $10^a - 10^b$ will be divisible by m for some natural numbers a and b. Suppose b is smaller than a, then the difference may be written

$$10^b(10^{a-b} - 1).$$

Since m divides this product and has no factors except 1 in common with 10^b it must divide $10^{a-b} - 1$, which is what we wanted to show. Thus we have shown that some power of 10 has a remainder of 1 when divided by m. Call k the smallest such power greater than zero.

From this we can conclude two things. First, the remainders when we compute the decimal expansion of $\frac{1}{m}$ will be just the remainders when the powers of 10 are divided by m. As soon as we get a remainder 1, the decimal begins to repeat and not before. So our number k is the number of digits in the repeating portion of the decimal.

Second, we may write any number in the form

$$a + b \times 10^k + c \times 10^{2k} + d \times 10^{3k} + \ldots$$

Since the remainder when 10^k is divided by m is 1, the remainder when our number is divided by m is the same as the remainder when

$$a + b + c + d + \ldots$$

is divided by m. We must notice, of course, that a, b, c, d are not digits in general but they are natural numbers less than 10^k.

How far the student can progress here, remains to be seen. Certainly he should not be pushed.

For casting out the elevens, instead of adding the digits, we would take the units digit, minus the tens digit, plus the hundreds digit and so forth. An example would be this:

637	7 - 3 + 6 = 10
× 152	2 - 5 + 1 = -2
product: 96,824	product: -20
4 - 2 + 8 - 6 + 9 = 13	0 + 2 = 2
3 - 1 = 2	

We come out with 2 in each case. This test is a little less likely to check when the answer is wrong but it is much more difficult to apply.

For the corresponding trick mentioned in Exercise 7, one cannot scramble the digits indiscriminately. The simplest direction to give would be to reverse the order of the digits and add the two numbers if the number of digits is even but subtract the two numbers when the number of digits is odd. This can be seen as follows: Suppose there are three digits a, b, c. Then the remainder for the given number would be that for $a - b + c$ and for its reverse $c - b + a$. Then the difference would have a zero remainder. If the number has four digits a, b, c, d, the sum for the given number would be $-a + b - c + d$ and for its reversal would be $-d + c - b + a$. Then the sum would have a zero remainder. Then it would not be sufficient to know the sum of all but one of the digits in the answer. You would have to ask for all the digits but the last, or something like that. This trick is much more complicated. Its only advantage is that it does not have the case of failure that that for nine has.

Exercises 3-2. Answers

1. (a) 226843

Sum of digits 25

$\frac{25}{9} = 2$ remainder 7

Not divisible by 9.

67945
Sum of digits 31
$\frac{31}{9}$ = 3 remainder 4
Not divisible by 9.

427536
Sum of digits 27
$\frac{27}{9}$ = 3 remainder 0
Divisible by 9.

45654
Sum of digits 24
$\frac{24}{9}$ = 2 remainder 6
Not divisible by 9.

(b) and (c) The remainder upon division by 9 is equal to the remainder when the sum of the digits is divided by 9.

2. (a) The uniqueness property of addition. (Refer to page 47 of the students' text.)
   (b) Yes

3. (a) Yes
   (b) Same reason as number 2.

4. (a) Yes (c) Yes
   (b) Yes (d) Yes

5. (a) $\frac{(9 + 1)^{20}}{9}$ has a remainder of 1^{20} or 1.

   (b) $\frac{(3 \times 3 + 1)^{20}}{3}$ has a remainder of 1^{20} or 1.

   (c) $10^{20} = 100^{10} = (99 + 1)^{10}$ has a remainder of 1^{10} or 1.

6. $\frac{(6+1)^{20}}{6}$ has a remainder of 1.

7. (a) divisibility by 4: If the number formed by the last two digits is divisible by 4, then the number is divisible by 4.
   (b) divisibility by 8: If the number formed by the last three digits is divisible by 8, then the number is divisible by 8.
   (c) divisibility by 25: If the number formed by the last two digits is divisible by 25 (e.g., 00, 25, 50, 75), then the number is divisible by 25.

8. 2, 3, 4, 6, 12 (0 is divisible by 12).

9. (a) multiples of powers of 7. (This includes negative powers of 7, e.g., $\frac{13}{49}$ in the decimal system is .16 in the system to the base 7.)
   (b) $(.12541\ \ldots)_7$

10. remainder 3.

11. (a) $\frac{(9+1)^{20}-1}{9}$ $1 - 1 = 0$ (Refer to Exercise 5.)
    (b) $\frac{(6+1)^{108}-1}{6}$ $1 - 1 = 0$ (as above)

12. Cast 9's from the sum as you go along. (See page 49 in students' text.)

Exercises 3-3. Answers

1.
```
   927   sum of digits: 18  sum of digits: 9  sum of digits: 9
   865   sum of digits: 19  sum of digits: 10 sum of digits: 1
  4635                                       Product          9
  5562
 7416
801855   sum of digits: 27  sum of digits: 9.
```

2. Answered in the text.

3. 810855.

4. 3, 0, 0.

5. Answered in the text.

6. Casting out the sixes.

7. 7543
   5437
   2106 sum of digits = 9

   $2 + 1 + 0 = 3$, other digit is 6,
   or $1 + 0 + 6 = 7$, other digit is 2,
   or $2 + 1 + 6 = 9$, other digit is 0 or 9.

Exercises 3-4-b. Answers

1. (1) a. $758 = 7 \times 10^2 + 58$ which has the remainder $7 + 3 = 10$ when divided by 11 since 100 has the remainder 1.

   b. $758 = 7 \times 10^2 + 5 \times 10 + 8$ which has the remainder $8 - 5 + 7 = 10$.

   (2) a. $7246 = 72 \times 10^2 + 46$ which has the same remainder as has $72 + 46$, that is, $6 + 2$ or 8.

   b. $7246 = 7 \times 10^3 + 2 \times 10^2 + 4 \times 10 + 6$ which has the same remainder as has $6 - 4 + 2 - 7 = -3$, that is, 8.

   (3) a. $81675 = 8 \times 10^4 + 16 \times 10^2 + 75$ which has the same remainder as has $8 + 5 + 9 = 22$. Hence the remainder is 0.

   b. $81675 = 8 \times 10^4 + 1 \times 10^3 + 6 \times 10^2 + 7 \times 10 + 5$ which has the same remainder as has $5 - 7 + 6 - 1 + 8 = 11$. Hence the remainder is zero.

2. Since 8 is 1 more than 7, a number to the base 7 can be tested for divisibility by 8 in the same way that we can test for divisibility by 11 in the decimal system. For instance, consider $(5326)_7$.

Using the first method we have $(5326)_7 = (53)_7 \times (10^2)_7 + (26)_7$. Since 8 is $(11)_7$, the remainder when $(10^2)_7$ is divided by 8 is 1. Thus the remainder when the given number is divided by 8 is the same as when $(53)_7 + (26)_7$ is divided by $(11)_7$. But $(53)_7 - (44)_7 = 6$ and $(26)_7 - (22)_7 = 4$ and hence the remainder is the same as when $6 + 4$ is divided by 8, that is, 2.

Using the second method we have $(5326) = 5 \times (10^3)_7 + 3 \times (10^2)_7 + 2 \times (10)_7 + 6$. When this is divided by 8 the remainder is the same as that for $6 - 2 + 3 - 5 = 2$.

3. (See page 20 of the teachers' commentary.)
3, 9, 27, 37, 3 × 37, 9 × 37, 999.

4. (See page 20 of the teachers' commentary.)
In the numeral system to the base seven, we can test for division by grouping in triples all the divisors of $7^3 - 1 = 342$. These divisors are:

2, 3, 6, 9, 18, 19, 38, 57, 114, 171, 342.

In the number system to the base twelve, we would have the divisors of $12^3 - 1 = 1727$. These factors are 11, 157, 1727.

5. (a) Yes. The remainders when the powers of 10 are divided by 11 are -1, 1 - 1, 1, ... which has a period of 2 since 11 is a divisor of $10^2 - 1$.

(b) The divisors 3 and 9 listed in the answers to Exercise 3 are divisors of $10^1 - 1$ as well as of $10^3 - 1$. All the others have three digits in the decimal equivalent of their reciprocals and for these, grouping the digits in threes gives a divisibility test.

6,7. See Teachers' Commentary, pages 22, 23.

	3	7	9	11	13	17	19	21	37	101	41
1	1	1	1	1	1	1	1	1	1	1	1
10^1	1	3	1	10	10	10	10	10	10	10	10
10^2	1	2	1	1	9	15	5	16	26	100	18
10^3	1	6	1	10	12	14	12	13	1	91	16
10^4	1	4	1	1	3	4	6	4	10	1	37
10^5	1	5	1	10	4	6	3	19	26	10	1
10^6	1	1	1	1	1	9	11	1	1	100	10
10^7	1	3	1	10	10	5	15	10	10	91	18
10^8	1	2	1	1	9	16	17	16	26	1	16
10^9	1	6	1	10	12	7	18	13	1	10	37
10^{10}	1	4	1	1	3	2	9	4	10	100	1
11^{11}	1	5	1	10	4	3	14	19	26	91	10
10^{12}	1	1	1	1	1	13	7	1	1	1	18
10^{13}	1	3	1	10	10	11	13	10	10	10	16
10^{14}	1	2	1	1	9	8	16	16	26	100	37
10^{15}	1	6	1	10	12	12	8	13	1	91	1
10^{16}	1	4	1	1	3	1	4	4	10	1	10

The number of digits in the cycle in the repeating decimal is the same as the number of digits in the cycle of remainders. Since, for example, there are five remainders in the column headed by 41, one can test for divisibility by 41 by grouping the digits in fives.

For the answers to questions on pages 57 and 58 look on pages 21 and 22 in the teachers' commentary.

SUPPLEMENTARY UNIT 4

OPEN AND CLOSED PATHS

The problem of the bridges of Königsberg (called Kaliningrad after the Russians took over in World War II) is one of the first problems ever solved in the field of topology. This is the branch of mathematics which deals with the properties of geometrical figures which are left unchanged by continuous one-to-one transformations. You may think of the geometry on a rubber sheet. What properties of the geometrical figures are not changed when you stretch or shrink the sheet in any way whatsoever, so long as you don't tear anything or glue parts together?

The object of this chapter is to show the children that there are interesting properties of geometric figures which do not involve any process of measurement. Section 3 is probably too difficult for any but the brightest pupils, and might be assigned to individuals as outside reading for extra credit. The other parts are for good, but not exceptional students, and may be suitable for homogeneously grouped classes, as well as for independent work for a few good students.

The problem of the existence of Hamiltonian paths, discussed in Section 4, is an excellent example of a question which any child can understand, but which has never been answered. You can use this to show the class that mathematics is a growing subject and that there is always more to do in mathematics.

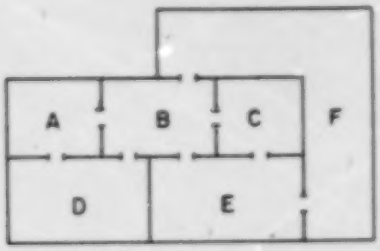
In Section 1 we developed the necessary conditions that there be, in a diagram, a path which goes over every bridge just once. (Incidentally, in the technical literature of graph theory the term "edge" is used instead of our term "bridge".) The main results are stated in Theorem 1 of Section 2.

At this level of maturity most children have difficulty with abstract general proofs. They do understand and appreciate "proofs" given in terms of concrete cases, treated by reasoning which they recognize as generally valid. The children will be

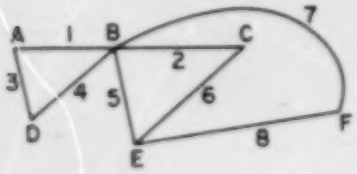
satisfied with the discussion of Theorem 1, although the mature student would want proof of Theorem 1 stated in general terms.

In the diagram of the Königsberg bridges, the vertex A is connected to 5 bridges, while each of the other three vertices is connected to 3 bridges. All 4 vertices are odd (see Section 2). Therefore there is neither an open nor a closed path which goes over each bridge just once.

You may give the children another concrete situation which leads to the same problem. Here is the floor plan of a house showing 6 rooms and 8 doorways. Is there a path which goes



through each doorway exactly once? We can make a diagram which



shows the rooms and their connections. In this diagram the vertices B and E are odd and the others are even. The path

B1A3D4B5E8F7B2C6E

goes through each doorway just once. If we insert a doorway connecting room C with room F, then the vertices C and F will also be odd. In that case, there is no path of the desired kind.

Exercises 1 to 4 are preparation for Section 3.

In the exercises of Section 2, the pupil is led to discover (in Exercise 2-2) empirically that the number of odd vertices in any diagram is even. The rest of these exercises lead the youngster to discover a proof of this fact.

In Exercises 2-2 and 2-3 you are not supposed to check all the diagrams the pupils make for themselves. The important thing is that they discover certain generalizations for themselves. If you teach the chapter to the whole class, you may have several pupils draw diagrams on the blackboard and have the whole class check the classification of the vertices as odd or even.

The only parts of Section 3 which might be suitable for more than a few students are the discussions of the distinction between necessary and sufficient conditions and the remarks on the importance of stating exactly what we mean.

Most pupils will overlook the fact that our definition of the term "diagram" does not require that a diagram be connected. It is instructive for them to learn, by bitter experience, not to jump to conclusions.

In 4-3 if Q_8 is not the same as Q_2, then the path is open. The point Q_8 is then an endpoint of the path. Then the path contains an odd number of bridges connected to Q_8 But, by assumption, the total number of bridges connected to Q_8 is even. Then the path cannot contain all of the bridges through Q_8 in the diagram.

The proofs of Theorem 2 and 3 are examples of <u>reductio ad absurdum</u>.

Exercises 4-1-a. Answers

1. (a) Open path

A 2 R 4 N 5 A 1 X 3 N or A 1 X 3 N 4 R 2 A 5 N or

A 5 N 3 X 1 A 2 R 4 N or A 5 N 4 R 2 A 1 X 3 N

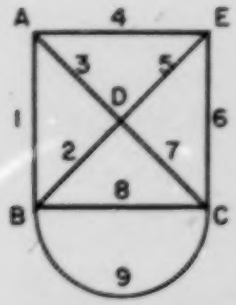
(b) Closed path

One possibility is: A 2 0 1 N 5 Y 3 0 7 R 6 A

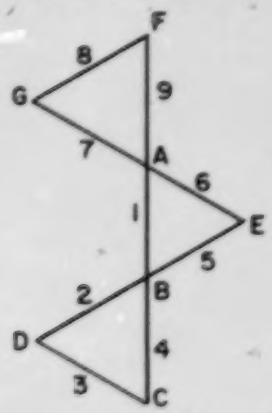
(c) Open path

One possibility is: A 6 X 4 C 2 D 5 X 3 B 1 A 7 D

2. (a) Open path: A 3 D 7 C 8 B 1 A 4 E 5 D 2 B 9 C 6 E.

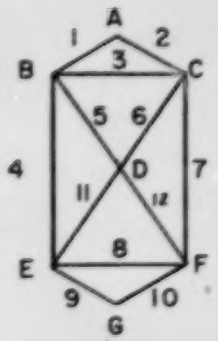


(b) Closed path A 1 B 2 D 3 C 4 B 5 E 6 A 7 G 8 F 9 A

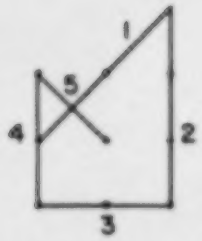


(c) Closed path. One possibility is:

A 1 B 5 D 12 F 8 E 9 G 10 F 7 C 6 D 11 E 4 B 3 C 2 A



3. One possibility is

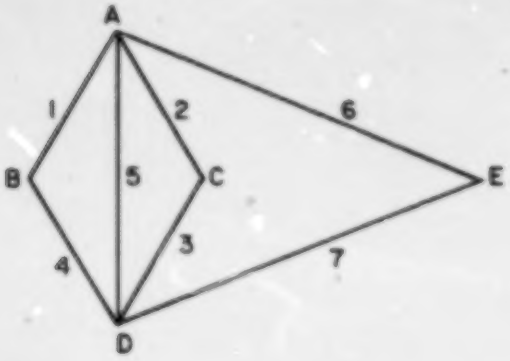


4. (a) No path is possible.

(b) An open path.
One possibility is A 4 B 5 C 6 D 2 E 1 A 3 D

(c) No path is possible.

(d) A closed path. One possibility is
A 6 E 7 D 3 C 2 A 1 B 4 D 5 A



Exercises 4-1-b. Answers

1.	odd vertices	even vertices	number of odd	number of even
I	A, B, E, G	C, D, F	4	3
II		all	0	12
III	A, C	all others	2	6
IV	A, E	all others	2	3

2. (a) I, III, IV.

   (b) I, II.

3. In II, the path
   A 1 B 2 C 3 D 4 E 5 F 6 G 7 H 8 I 9 J 10 K 11 L 18 -
   B 13 D 14 F 15 H 16 J 17 L 12 A.
   In III, the path.
   A 1 B 6 H 10 G 5 B 4 F 8 E 3 B 2 C 11 D 14 G 9 F 13 -
   D 12 E 7 C.
   In IV, the path
   A 1 B 9 C 2 A 7 D 3 B 5 E 6 C 4 D 8 E.
   In each of these there are also many other paths.

4. I. B 2 C 6 D 3 B 4 E 8 F 5 B
   II. B 1 A 12 L 11 K 10 J 9 I 8 H 7 G 6 F 5 E 4 D 3 C 2
   B 13 D 14 F 15 H 16 J 17 L 18 B
   III. B 2 E 7 C 11 D 12 E 8 F 13 D 14 G 10 H 6 B 5 G 9 F 4 B.
   IV. B 9 C 2 A 7 D 4 C 6 E 8 D 3 B

5. A 1 B 2 C 5 D 4 A 3 C and A 4 D 5 C 3 A 1 B 2 C

Exercises 4-2. Answers

1. See above table.

2. The number of odd vertices in any diagram is even.

3.

degree	Number of vertices	I	II	III	IV
1		2	0	1	0
2		2	6	1	0
3		1	0	1	2
4		1	6	4	3
5		1	0	0	0
6		0	0	1	0

4. $V = V_1 + V_2 + V_3 + \ldots$

6. Two, one for each end of the bridge. The number of pairs is $2 \cdot B$.

7. Each vertex of degree k occurs in k pairs. The total number of pairs in which a vertex of degree k occurs is $k \cdot \frac{1}{2}$.

8. $V_1 + (2 \cdot V_2) + (3 \cdot V_3) + \ldots$

9. $V_1 + V_3 + V_5 + \ldots$

10. $(2 \cdot B) - U = (V_1 + (2 \cdot V_2) + (3 \cdot V_3) + \ldots)$
$- (V_1 + V_3 + V_5 + \ldots) = (2 \cdot V_2) + (2 \cdot V_3) + (4 \cdot V_4)$
$+ (4 \cdot V_5) + \ldots$.

11. By the formula, we have
$U = (2 \cdot B) - [(2 \cdot V_2) + (2 \cdot V_3) + \ldots]$
$= 2 \cdot [B - [V_2 + V_3 + (2 \cdot V_4) + (2 \cdot V_5) + \ldots]]$,
which is even.

Exercises 4-3. Answers

1. (a) No.

(b) No.

2. BRAINBUSTER. Let A and B be the two odd vertices. Consider the longest path from A to B which does not go over any bridge more than once. Color this path blue. Continue as in the proof of Theorem 2, Section 4.

Exercises 4-4. Answers

None of the diagrams in this section has a Hamiltonian path.

SUPPLEMENTARY UNIT 5

FINITE DIFFERENCES

A basic method of finding polynomial formulas to fit tables of values is that called "finite differences." Here the student is given a very brief introduction to this theory. The teacher is referred to two books on the subject:

1. C. H. Richardson "An Introduction to the Calculus of Finite Differences" Van Nostrand (1954) 142 pages.
2. A book on the same subject by F. H. Miller is to be published by Henry Holt and Company.

These two books are too advanced for seventh or eighth graders - in fact for most high school students - but they should be intelligible to anyone who has had a first course in calculus. This does not mean that calculus forms the basis for finite differences, but merely that the analogies between the formulas in the two subjects are very striking.

Though an attempt is made to keep the algebra used in this unit to a minimum some is certainly needed; for instance the expansion of $(n + 1)^3$ is needed. Of course in this unit as in the other supplementary ones, the student should do a large body of experimentation himself and be given the opportunity to make guesses and verify them.

5-1. Arithmetic Progressions

Here the arithmetic progression is taken up from the standpoint of finite differences and the relationship with the triangular numbers is exploited. The student should see both the geometrical and algebraic arguments and how they are related. Though in the exercises a basis is laid for the general formula for the sum of an arithmetic progression, more stress is laid on finding the sum than either in getting or using the formula.

The most important idea of the section is stated formally in the theorem in Section 3. In all the exercises and examples except Problem 15 the entries in the table are increasing. Possibly more examples should be given in which the differences are negative.

In Table II the entries might be extended one to the left with a zero in row 1 and a 1 in row 2. Then the entries in the second row of Table II would be just the entries in the first row of Table I.

The pupil might be interested to notice that the triangular numbers are those on the second slanting line of the Pascal triangle. The pyramidal numbers defined in Problem 3 of Exercises 5-3b, are those in the third slanting line of the Pascal triangle

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5-2. More Sequences

There are two methods for finding the formula for the sum of the first n squares. Both rest on the fundamental fact that in any table, the n-th term minus the first term, is the sum of the first differences to the left of the n-th term. This section works downward in the table and finds the formula for the sum of squares by starting with the cubes. (The second method, that in Section 3, works upward in the table and establishes the formula by guessing what form it should take and determining the appropriate coefficients.)

By way of introduction here, the formula for the sum of the first n integers is gotten from Table III, the table of squares. This is a more complex way of doing it but it does illustrate the method which will be used in the latter part of the section.

Gradually there should emerge the idea that a linear formula fits a table of numbers in an arithmetic progression, a quadratic formula if the first differences are in an arithmetic progression, a cubic formula if the second differences are in an arithmetic progression, etc.

Also in the exercises of the sections the idea is developed that a linear formula can be found to fit any two numbers since any two numbers can be used to start an arithmetic progression; a quadratic formula can be used to fit any three numbers since any three numbers can be used for the first three entries in a table whose first differences form an arithmetic progression, etc.

5-3. Finding Formulas that Fit

Here the method is used which guesses at a formula for the table and then determines the constants by means of the first and second differences.

The product of any three consecutive integers must be divisible by 6 since at least one is even and at least (in fact, only) one is divisible by 3. For four consecutive integers, one must be divisible by 2, one by 4 and one by 3; in other words the product must be divisible by 24. A bright student might like to see what happens for five consecutive integers.

Triangular, square, pentagonal, etc. numbers are called "figurate numbers" or "polygonal numbers." The general formula for the n-th k-gonal number is

$$(k - 2) \times \frac{n^2 - n}{2} + n.$$

That is, when k is 3 we have triangular numbers, k = 4 gives square numbers, k = 5 gives pentagonal numbers, etc. Notice that they are all quadratic expressions and hence the first differences are in an arithmetic progression.

Of course one may have numbers associated with pyramids whose bases are closed square regions, closed pentagonal regions, etc.

Exercises 5-1. Answers

1. (a) 15, 16, 17, ... 32, 33, 34.

   (b) $(14 + 1)$, $(14 + 2)$, $(14 + 3)$, ... $(14 + 18)$, $(14 + 19)$, $(14 + 20)$.

   (c) $(14 + 14 + 14 \ldots) + (1 + 2 + 3 \ldots)$.

   (d) $(20 \cdot 14) + \frac{1}{2} \cdot 20 \cdot 21)$

   (e) 490

2. (a) 142, 143, 144, ... 149, 150, 151.

   (b) $(141 + 1)$, $(141 + 2)$, $(141 + 3)$ ... $(141 + 8)$, $(141 + 9)$, $(141 + 10)$.

   (c) $(141 + 141 + 141 \ldots) + (1 + 2 + 3 \ldots)$

   (d) $10 \cdot 141 + \frac{1}{2} \cdot 10 \cdot 11$

   (e) 1465

3. $nb + \frac{1}{2}n(n + 1)$ or $n(b + \frac{1}{2}n + \frac{1}{2})$ or $2n(2b + n + 1)$

4. 13, 15, 17, ... 47, 49, 51

   $(11 + 2)$, $(11 + 4)$, $(11 + 6)$, ... $(11 + 36)$, $(11 + 38)$, $(11 + 40)$.

   $20 \cdot 11 + 2 \cdot (1 + 2 + 3 \ldots)$

   $220 + (2 \cdot \frac{1}{2} \cdot 20 \cdot 21)$

   $220 + 420 = 640$.

5. $n \cdot b + n(n + 1) = n(b + n + 1)$ or $n(n + b + 1)$

6. The next three entries are: 21, 23, 25. The differences are all the coefficient of n in $2n + 7$.

7. For $3n + 7$ the first five values are: 10, 13, 16, 19, 22 and the differences are 3, the coefficient of n; for $2n + 6$ the first five values are 8, 10, 12, 14, 16 and the differences are 2, the coefficient of n.

8. The differences are 5. In fact, if the form is $an + b$, the next term would be $a(n + 1) + b$ and the differences are all a. This is leading up to the theorem in Section 3.

9. Using the trick used at the end of Section 1, we write

$$\begin{array}{ccccccccc} 1 & & 3 & & 5 & \ldots & 2n-5 & 2n-3 & 2n-1 \\ 2n-1 & & 2n-3 & & 2n-5 & & 5 & 3 & 1 \end{array}$$

and the sum is $\frac{1}{2}(2n)n = n^2$. In the formula developed in Problem 5 b must be $^-1$ to have the first number in the sequence be 1; and

$$\begin{aligned} s &= n(n + {}^-2 + 1) \\ &= n \cdot n \\ &= n^2 \end{aligned}$$

10. Replacing n by n - 1 in the formula for the n-th triangular number, we get $\frac{1}{2}(n - 1)(n - 1 + 1) = \frac{1}{2}n(n - 1)$.

11. We can use the trick of Problem 6 to get

$$\frac{1}{2}(1 + 1 + nd)(n + 1) = \frac{1}{2}(2 + nd)(n + 1)$$

or it can be done by writing the sum:

$$(n + 1) + d(0 + 1 + 2 + \ldots + n).$$

12. In the above replace n by n - 1 to get

$$\frac{1}{2}n[2 + (n - 1)d].$$

13. Add (b - 1)n to the previous answer to get

$$\frac{1}{2}n[2b - 2 + 2 + (n - 1)d] = \frac{1}{2}n[2b + (n - 1)d].$$

14. $$\begin{array}{ccccccccc} & & 7 & & 12 & & 17 & & 22 & & 27 \\ & & & 5 & & 5 & & 5 & & 5 & \end{array}$$

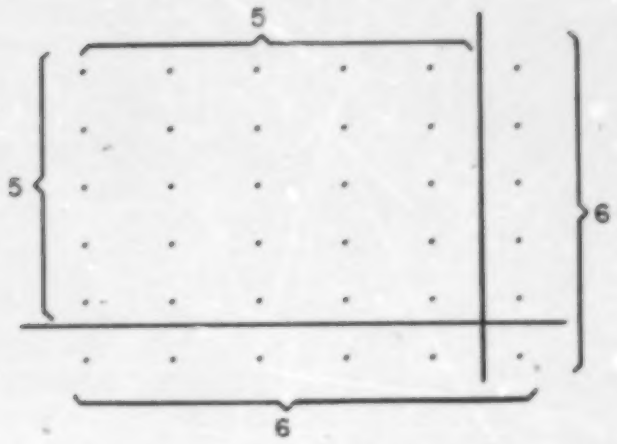
We add 5 each time.

15. 7 5 3 1 $^-1$ ... Here we continue to subtract 2.

16. To construct such a table, compute the first difference from the given two numbers and then add this difference to each number of the table to get the next.

Exercises 5-2. Answers

1. (a)



(b) To get a square of $(n + 1)$ dots on a side from one with n dots on a side, we can add two edges of n dots each and a 1 in the corner, which accounts for the $2n + 1$.

2. The methods of the section may be used or the sum can be made to depend on that already found as follows:

$$2^2 + 4^2 + 6^2 + \ldots + (2n)^2$$
$$= 4 \cdot 1^2 + 4 \cdot 2^2 + 4 \cdot 3^2 + \ldots + 4 \cdot n^2$$
$$= 4 \cdot \frac{2n^3 + 3n^2 + n}{6} = \frac{2}{3} \cdot (2n^3 + 3n^2 + n).$$

3. Again the methods of the section may be used or the following:

$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2$$
$$= (2 \cdot 1 - 1)^2 + (2 \cdot 2 - 1)^2 + \ldots (2n - 1)^2$$
$$= (4 \cdot 1^2 - 4 \cdot 1 + 1) + (4 \cdot 2^2 - 4 \cdot 2 + 1) + \ldots$$
$$(4n^2 - 4n + 1) = 4s - 4 \cdot \tfrac{1}{2}(n^2 + n) + n$$
$$= (\tfrac{2}{3})(2n^3 + 3n^2 + n) - 2n^2 - 2n + n = \tfrac{1}{3} \cdot (4n^3 - n).$$

4.

4	7	12	19	28	39
3	5	7	9	11	
2	2	2	2		

where we first write the third row, then the second and then the first.

5.

10	5	11	28	56
-5	6	17	28	
11	11	11		

6. Starting with any three numbers we could construct such a table if we get the first two first differences and the first second difference and then compute the table from the bottom up.

*7. Here we would use the table of fourth powers and first compute the difference between the $(n + 1)$st and the n-th fourth powers, getting $4n^3 + 6n^2 + 4n + 1$ as the n-th term in the set of first differences. Taking the sum, as we did above, we have

$$4 \times (\text{sum of cubes}) + 6 \times (\text{sum of squares}) + 4 \times (\text{sum of integers}) + n = (n + 1)^4 - 1.$$

Let C stand for the sum of the first n cubes and use the formula above the sum of squares to get

$$4C + 6 \times \frac{1}{6}(2n^3 + 3n^2 + n) + 2n(n + 1) + n = n^4 + 4n^3 + 6n^2 + 4n.$$

Solving for C, we get

$$C = \frac{1}{4}(n^4 + 2n^3 + n^2) = [\frac{1}{2}n(n + 1)]^2.$$

Notice that the formula for C is the square of the formula for the sum of the first n integers.

Exercises 5-3-a. Answers

1. (a) 6 12 20 30 42

   (b) 0 1 10 20

   Except for the entry 0, these numbers are one-sixth of those in the table for $n(n + 1)(n + 2)$. The reason is that $n^3 - n(n - 1)n(n + 1)$ which is the product of three consecutive integers whose middle one is n. The result seems more interesting if one notices that this means that if a number is subtracted from its cube, the result is the product of three consecutive numbers of which the middle one is the given number. (Here "number" means integer.)

   (c) 2 10 30 68 130

2. Since $n^4 - n^2$ is of the fourth degree, the third differences should be in an arithmetic progression. To check this write

0	12	72	240	600	1260
	12	60	168	360	660
		48	108	192	300
			60	84	180
				24	24

Exercises 5-3-b. Answers

1. (a) The first differences are 5 and hence the numbers are in arithmetic progression of the form $5n + b$. But $5 + b = 2$ which implies $b = -3$ and the formula is $5n - 3$.

   (b) Here the first and second differences are

$1\frac{1}{4}$	$2\frac{1}{4}$	$3\frac{1}{4}$	$4\frac{1}{4}$
	10	10	10

   Hence the first differences are of the form $10n + b$ and since the first one is $1\frac{1}{4}$, $b = \frac{1}{4}$. So the n-th first difference is $10n + \frac{1}{4}$.

The formula to fit the table should be like: $an^2 + bn + c$ and, as in the section we have

$$10n + 4 = 2an + (a + b).$$

Thus $a = 5$, $b = -1$ and $5 = 5 \times 1^2 - 1 \times 1 + c$, implies $c = 1$. So the formula is $5n^2 - n + 1$.

(c) Here the first and second differences are:

12 18 24 30

6 6 6

Here the formula for the first differences is $6n + 6$ and

$$6n + 6 = 2an + (a + b)$$

implies $a = 3$, $b = 3$. Then $8 = 3 \times 1^2 + 3 \times 1 + c$ implies $c = 2$ and the formula is $3n^2 + 3n + 2$.

2. The first, second and third differences are

8 20 38 62 92

12 18 24 30

6 6 6

Notice that the first differences here are the numbers in table 1(c). (If this were not the case, we would compute the formula for the first differences as we did above.) Hence the formula for the n-th first difference is $3n^2 + 3n + 2$.

Since the second differences form an arithmetic progression we guess that the formula for the numbers of the table must be $an^3 + bn^2 + cn + d$ for proper choices of a, b, c, and d.

3. The n-th pyramidal number is the sum of the first n triangular numbers. By the same method as that above, we can get the formula $\frac{1}{6}(n^3 - n)$ which is the formula for the table in Problem 1(b) in Exercises 3-a.

4. A polynomial of the fourth degree.

5. This is merely a matter of checking numerical values. Anyone interested in learning more about polygonal numbers should look in Volume 2 of "The History of the Theory of Numbers" by L. E. Dickson.

6. In each case the first differences repeat the table except for the initial entry in (b). Suppose the entries were

$$a \quad b \quad c \quad d \quad \ldots$$

If the first differences gave the same table we would have

$$a = b - a, \quad b = c - b, \ \ldots$$

and since the first equation implies $2a = b$, and similarly for the others, the entries would have to be

$$a \quad 2a \quad 4a \quad 8a \quad \ldots$$

in geometric progression.

If the second first difference were the first term of the table, we would have $c - b = a$, that is, $c = b + a$. So if this property were to continue throughout the table, each entry would have to be the sum of the previous two. The student can carry this further if he is interested.

7. An arithmetic progression has a formula $an + b$ associated with it.

8. and 9. See pages 38 and 39 of this commentary.

SUPPLEMENTARY UNIT 6

RECENT INFORMATION ON PRIMES

The main purpose of this unit is to give the children some idea of mathematics as a living, growing science. We try to drive this home by relating Supplementary Unit 6 to a paper published in October, 1958, thus emphasizing that even today mathematical discoveries which they can understand are being made. A minor goal is to show how mathematical theory is used in work with modern computers. You may stress that the computer is a giant idiot slave which does exactly what you tell it to do, neither more nor less, and that someone with brains still must give the instructions to the computer.

In computing the number of seconds in a year, teach the children that we do not want to know the exact number, but only the order of magnitude. We therefore round off the numbers freely in order to simplify our work. Ask, "How many seconds are there in a minute? How many minutes are there in an hour? How many seconds are there, then, in an hour?" Do not carry out the calculation, just indicate it:

60×60 seconds in an hour.

"How many hours are there in a day, and how many days in a year?" Show the total number of seconds in a year:

$60 \times 60 \times 24 \times 365$ seconds in a year.

Even here we are approximating, since a year is about $365\frac{1}{4}$ days. We can round off thus:

$60 \times 60 = 3{,}600 = 4{,}000$ approximately,

$4{,}000 \times 24 = 4{,}000 \times 25 = 100{,}000$ approximately, and

$100{,}000 \times 365 = 100{,}000 \times 400 = 40{,}000{,}000$ seconds in a year,

or $4 \cdot 10^7$, approximately. The actual number is 31,536,000, ignoring the error of $\frac{1}{4}$ of a day. The important thing is that

this is of the order of 10^7, or an 8 digit number in decimal notation.

The number of years in 10^{97} seconds is

$$\frac{10^{97}}{4\cdot 10^7} = \frac{10\cdot 10^{96}}{4\cdot 10^7} = (2.5)\cdot 10^{89},$$

which is a 90 digit number. Have the children look up the geologists' estimate of the age of the earth. See, for example, Gamow's book, the Biography of the Earth.

You may review the symbols for inequality:

$\neq$ is unequal to,

$<$ is less than,

$>$ is greater than,

$\leq$ is equal to or less than,

$\geq$ is equal to or greater than.

It is helpful, in keeping straight the meanings of "<" and ">", to remind the children that the smaller end of the symbol points to the smaller number.

At the rate of one division per .001 of a second, it would take about $3\cdot 10^{39}$ years to perform 10^{50} divisions.

Discussion of Exercise 6-1

1.

m	1	2	3	4	5	6	7	8	9	10	11	12
n	1	3	7	15	31	63	127	255	511	1023	2047	4095

m	13	14	15	16	17	18	19	20
n	8191	16383	327677	65535	131071	262143	524287	1048575

2. (a) 3 divides 3, 15, 63, 255, etc.

(b) 7 divides 63, 511, 4095, etc.

(c) 31 divides 31, 1023, 32767, and 1048575.

(d) If m is divisible by b then n is divisible by $2^b - 1$.

Note the difference between

$$3(2^4) = 3^{16},$$

$$(3^2)^4 = 3^8.$$

6-2 Proth's Theorem

If m is odd, and k = 1 then $n = 2^m + 1$ is divisible by 3. One way to see this is to notice that the successive odd numbers differ by 2. Suppose m is odd, and we already know that $2^m + 1$ is divisible by 3. The next number of this type is

$$\begin{aligned} 2^{m+2} + 1 &= (2^m \cdot 2^2) + 1 \\ &= (4 \cdot 2^m) + 1 \\ &= ((3 + 1) \cdot 2^m) + 1 \\ &= (3 \cdot 2^m) + (1 \cdot 2^m) + 1 \\ &= (3 \cdot 2^m) + (2^m + 1). \end{aligned}$$

Since both numbers in this sum are divisible by 3, then so is the sum. Therefore, if $2^m + 1$ is divisible by 3, then so is $2^{m+2} + 1$. In other words, if the statement is true of one odd number m, then it is also true of the next one. But it is true for m = 1 ($2^1 + 1 = 2 + 1 = 3$, which is divisible by 3). Therefore it is true for m = 3. Therefore it is true for m = 5, etc.

It is, then, sufficient to test $n = 2^m + 1$ for primeness only when m is even.

Disscussion of Two Additional Examples.

6. When $k = 1$ and $m = 4$, we have $n = 2^4 + 1 = 17$ and

$$3^{\frac{17-1}{2}} + 1 = 3^8 = 6562.$$

Since 17 divides 6562, we see that 17 must be a prime according to the test, and it is! When $m = 6$, we have $n = 2^6 + 1 = 65$ and

$$3^{\frac{65-1}{2}} + 1 = 3^{32} + 1,$$

which is not divisible by 65. According to the test 65 is not a prime, which also checks. If you divide $3^{32} + 1$ (a sixteen digit number) by 65, the remainder is 62.

7. If $k = 5$ and $m = 2$ or 4, then n is divisible by 3. If $k = 5$ and $m = 3$, then $n = 41$ and

$$3^{\frac{41-1}{2}} + 1 = 3^{20} + 1,$$

which is divisible by 29. If $k = 7$ and $m = 4$, then $n = 113$ and

$$3^{\frac{113-1}{2}} + 1 = 3^{56} + 1,$$

which is a number of 27 digits.

It will probably be too much work for the children to divide this number by 113 directly. A better way is this. First calculate

$$3^5 = 243$$

and divide by 113. The remainder is 17. Therefore we have

$$3^5 = 17 + (113 \cdot \text{something}).$$

If we square 3^5, we obtain

$$3^{10} = 17^2 + (113 \cdot \text{something}),$$

or

$$3^{10} = 63 + (113 \cdot \text{something}),$$

since the remainder on dividing $17^2 (= 289)$ by 113 is 63. Proceeding in the same way we obtain

$$\begin{aligned} 3^{20} &= 63^2 + (113 \cdot \text{something}) \\ &= 14 + (113 \cdot \text{something}), \end{aligned}$$

and

$$\begin{aligned} 3^{40} &= 14^2 + (113 \cdot \text{something}) \\ &= 83 + (113 \cdot \text{something}), \end{aligned}$$

and

$$\begin{aligned} 3^{50} = 3^{10} \cdot 3^{40} &= (63 + \ldots)(83 + \ldots) \\ &= (63 \cdot 83) + (113 \cdot \text{something}) \\ &= 31 + (113 \cdot \text{something}), \end{aligned}$$

and

$$\begin{aligned} 3^{55} &= 3^5 \cdot 3^{50} \\ &= (17 \cdot 31) + (113 \cdot \text{something}) \\ &= 75 + (113 \cdot \text{something} \end{aligned}$$

and

$$3^{56} = 225 + (113 \cdot \text{something}).$$

But $225 + 1$ is divisible by 113 so that the test works!

This calculation is based on the equation

$$(a + c)(b + d) = (a \cdot b) + (a \cdot d) + (b \cdot c) + (c \cdot d).$$

If c and d are divisible by 113, then so is the sum of the last three products on the right.

The same method can be applied to other problems, in particular to the preceding exercises. Thus to test $3^{20} + 1$ for divisibility by 41, we may calculate as follows:

$$\begin{aligned} 3^4 &= 81 = 40 + (41\cdot\text{something}), \\ 3^8 &= 40^2 + (41\cdot\text{something}), \\ &= 1600 + (41\cdot\text{something}), \\ &= 1 \quad + (41\cdot\text{something}), \\ 3^{16} &= 1^2 \quad + (41\cdot\text{something}), \\ &= 1 \quad + (41\cdot\text{something}), \end{aligned}$$

and

$$\begin{aligned} 3^{20} &= 3^4\cdot 3^{16} = (40\cdot 1) + (41\cdot\text{something}) \\ &= 40 + (41\cdot\text{something}) \end{aligned}$$

so that $3^{20} + 1$ is a multiple of 41 since $40 + 1$ is divisible by 41.

At the rate of a thousand divisions per second, it would take about $3\cdot 10^{280}$ years to perform 10^{291} divisions.

8. In order to estimate 2^m yourself, you may use the fact that

$$\log_{10} 2 = .30103,$$

approximately, so that with only a small error

$$\begin{aligned} 2^{3217} &= 10^{(.30103)\cdot 3217} \\ &= 10^{96841}, \end{aligned}$$

which is a number of 969 digits in decimal notation. Similarly the number of digits in $2^m - 1$ is 687 when $m = 2281$, and 664 when $m = 2203$.

You can astound the children by calculating mentally, using

0.3 as an approximation for the logarithm of 2. You may look at a number like

$$3^{3217} - 1$$

for a few seconds, and say casually, that this number has about 966 digits ((.3)·3217 = 965.1). Similarly you may use the approximations

$$\log_{10} 3 = .47712 = .5, \quad \text{approximately,}$$

$$\log_{10} 5 = .69897 = .7, \quad \text{approximately,}$$

$$\log_{10} 7 = .84510 = .8, \quad \text{approximately,}$$

Thus 5^{1000} is a number of about 700 digits since (.7) 1000 = 700.

By using the method in the text, the children might proceed as follows:

$$2^{10} > 10^3,$$

therefore

$$\begin{aligned} 2^{3217} &> 2^{3210} = (2^{10})^{321} \\ &> (10^3)^{321} = 10^{963}, \end{aligned}$$

so that when m = 3217, n has at least 964 digits. On the other hand, we have

$$2^{13} < 10^4,$$

therefore

$$\begin{aligned} 2^{3217} &= 2^{3211} \cdot 2^6 \\ &= (2^{13})^{247} \cdot 2^6 \\ &< (10^4)^{247} \cdot 64 \\ &< 64 \cdot 10^{988}, \end{aligned}$$

so that n has at most 990 digits.

SUPPLEMENTARY UNIT 7

GAMES

Introduction

Some teachers who have taught or reviewed the seventh grade course of the SMSG have been concerned over the omission of the business applications which have been traditionally included in the seventh and eight grade mathematics course. The author of this unit, along with many other teachers of mathematics, feels that seventh graders are not naturally interested in such topics. If we are to teach such topics, we must motivate the students by the inclusion of some material which will be intriguing to students of this age. Children this age are interested in playing a great variety of games. Some of these games are like the games included in this unit. This study of game theory leads very naturally into a discussion of problems associated with a business. It is not the author's intention that students should develop a skill in the analysis of games. This part of the unit is intended to acquaint students with a very interesting new branch of mathematics and to motivate the study of the solution of typical business problems.

Game theory was systematically presented as a new branch of mathematics by John von Neuman in cooperation with Morgenstern in a book titled: The Theory of Games and Economic Behavior, published in 1928. Game theory is being used to analyze war strategy, economic behavior, and most other activities that involve decision making. The title, "game theory," was given to this branch of mathematics because the techniques used were originally developed from an analysis of games which were of varying degrees of complexity. Some of the very familiar games, such as certain card games, cannot be analyzed by present techniques. This is a young branch of mathematics and has many problems which are still unsolved.

Suggestions for the Teaching of this Unit

This unit offers good opportunity for students to see how much help mathematics can be in as simple a situation as the playing of the first game. You may wish to divide your class into couples and tell them the rules of the game, allowing them to play this game without the help of the analysis presented in the first section. Some of your students may discover the best strategy on their own. After playing this game, lead the students in a reading of the introduction and a thorough study of the section titled "Strategy". The answers to some of the questions in the material appear a few lines later, but your students should be encouraged to answer each question as it appears.

Game theory requires that we always consider the worst possible results. This is the reason Tom is concerned about the minimum number of points he will win and Jim is concerned about the maximum number of points he could lose in the first series of calls. Tom tries to keep the minimum number of points as high as possible. Jim looks for the least of the maximum losses to protect his game. This type of game, one having a saddle-point, is the simplest possible game we can analyze. For a game which does not have a saddle-point, a mixed strategy must be used. The calculation of the proper mixture of calls requires a knowledge of probability, therefore the author did not attempt to analyze such games. More information on this latter type of game analysis may be obtained from the book by Williams, The Compleat Strategyst.

The material in the Business Strategy Section is similar to the material usually taught to seventh graders. Following a discussion of the various terms used by businessmen there is an adequate number of problems involving various situations to reinforce an understanding of the material. It is important that in this section, the students again answer each question as they come to it in their reading. The only significant difference

to be found in the terminology used here is that "loss" is spoken as a "negative profit." Students may be reminded of temperatures below zero as another example of negative numbers used in some measurement.

The last section of the unit called "Payoff Matrix for a Business Decision," combines the techniques learned in Sections 7-1 and 7-2. This section could probably be taught best by having the whole class work together on both problems. The suggested forms should help students to organize their work. The payoff matrix which is associated with the last problem is a matrix which does not have a saddle-point, therefore the final solution of that problem cannot be computed. It is intended that this problem should stimulate the students to do further study of this topic when they have the appropriate mathematical background.

Key

7-1 Strategy

		Jim's Choice A	Jim's Choice B	
Tom's Choice	A	6	5	5
	B	5	4	4
		6	5	

The maximum payoff is least for the call "B".

Exercises 7-1. Answers

A. 1. (a) 4

(b) 2

2. 4, call "A".

3. (a) 7

(b) 4

4. 4, call "B".

5. Yes, Yes

6. Tom should always call "A".
   Jim should always call "B".

B. 1. 6, call "B".

2. 6, call "A".

3. Yes, Yes

4. Tom should always call "B".
   Jim should always call "A".

C. 1. 6, call "B".

2. 6, call "B".

3. Yes, Yes

4. Tom should always call "B".
   Jim should always call "B".

D. 1. (a) 2
   (b) 6
   (c) 3

2. 6, call "B".

3. (a) 11
   (b) 6
   (c) 9

4. 6, call "B".

5. Yes, Yes

6. Tom should always call "B".
   Jim should always call "B".

7-2 Business Strategy

Bike-margin $25

profit $10

For expenses = $30, the profit = -$5.00.

Exercises 7-2. Answers

Dime Store:

1. $90
2. $30
3. -$10

Ice Cream Man:

4. $23
5. $5.40
6. $17.60

Theatre:

7. $380
8. $270
9. $110

Service Station

10. $171.60
11. $46.80
12. $22.30
13. -$3.50

7-3 Payoff Matrix for a Business Decision

Vendor's order	Weather: warm	Weather: cold	
coffee	$15	$35	15
popcorn	$20	$30	20
	20	35	

Coffee
margin $35
profit $35

Popcorn
margin $30
profit $30

The least of the maximum "losses" for weather (30) equals the greatest of the minimum profits for the vendor ($200), therefore the game has a saddle-point and the vendors best strategy is always to order popcorn.

It is important to emphasize the point that the fundamental assumption here is that the weather is completely unreliable. Let w be the number of days it is warm and c the number of days it is cold. Then the profit for the season if the vendor sold only coffee would be

$$15w + 35c.$$

If the vendor sold only popcorn, the profit would be

$$20w + 30c.$$

Now the profit would be greater for coffee if

$$15w + 35c > 20w + 30c$$

that is, if

$$5c > 5w$$

or

$$c > w.$$

Thus if the vendor knew that there would be more cold days than hot ones he should sell coffee all the time for maximum seasonal profit. This is the opposite conclusion from that in the case when nothing is assumed known about the weather for the season.

Exercises 7-3. Answers

A. Completion of the chart.

		Margin	Profit
Hot dogs	hot day	$ 6.00	$ 4.00
	cold day	$10.00	$ 8.00
Soda Pop	hot day	$14.00	$12.00
	cold day	$11.00	$ 9.00
Ice Cream	hot day	$25.00	$23.00
	cold day	$ 9.00	$ 7.00

Weather

	Hot	Cold	
Hot Dogs	$ 4.00	$8.00	$4.00
Soda Pop	$12.00	$9.00	$9.00
Ice Cream	$23.00	$7.00	$7.00
	$23.00	$9.00	

There is a saddle-point therefore the vendor should always order soda pop. His profit will always be $9.00.

B. The four charts are:

	Ice Cream		Coffee
Sold on hot day	400		200
Total Receipts		$60.00	
Ordered for hot day	400		200
Total Costs		$28.00	
Margin		$32.00	
Operating expense		$ 3.00	
Profit		$29.00	

	Ice Cream		Coffee
Sold on hot day	200		400
Total Receipts		$60.00	
Ordered for cold day	200		500
Total Costs		$30.00	
Margin		$30.00	
Operating Expense		$ 3.00	
Profit		$27.00	

	Ice Cream		Coffee
Sold on cold day	200		500
Total Receipts		$70.00	
Ordered for cold day	200		500
Total Costs		$30.00	
Margin		$40.00	
Operating Expense		$ 3.00	
Profit		$37.00	

	Ice Cream		Coffee
Sold on cold day	200		200
Total Receipts		$40.00	
Ordered for hot day	400		200
Total Costs		$28.00	
Margin		$12.00	
Operating Expense		$ 3.00	
Profit		$ 9.00	

		Weather	
		hot	cold
Ordered For	Hot day	$29	$ 9
	Cold day	$27	$37

Here there is no saddle-point.